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LOGARITHMIC BUMP CONDITIONS AND THE TWO-WEIGHT BOUNDEDNESS OF CALDERÓN–ZYGMUND OPERATORS

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ABSTRACT. We prove that if a pair of weights (u, v) satisfies a sharp A_p-bump condition in the scale of all log bumps or certain loglog bumps, then Haar shifts map $L^p(v)$ into $L^p(u)$ with a constant quadratic in the complexity of the shift. This in turn implies the two weight boundedness for all Calderón-Zygmund operators. This gives a partial answer to a long-standing conjecture. We also give a partial result for a related conjecture for weak-type inequalities. To prove our main results we combine several different approaches to these problems; in particular we use many of the ideas developed to prove the A_2 conjecture. As a byproduct of our work we also disprove a conjecture by Muckenhoupt and Wheeden on weak-type inequalities for the Hilbert transform. This is closely related to the recent counterexamples of Reguera, Scurry and Thiele.

1. INTRODUCTION

In this paper we prove several partial results related to a pair of long-standing conjectures in the theory of two-weight norm inequalities. To state the conjectures and our results we recall a few facts about Orlicz spaces; see [3, Chapter 5] for complete details. Given a Young function A, the complementary function $\bar{A}$ is the Young function that satisfies

$$t \le A^{-1}(t)\bar{A}^{-1}(t) \le 2t, \qquad t > 0.$$

We will say that a Young function $\bar{A}$ satisfies the $B_{p'}$ condition, $1 < p < \infty$, if for some $c > 0$,

$$\int_c^\infty \frac{\bar{A}(t)}{t^{p'}} \frac{dt}{t} < \infty.$$

If A and $\bar{A}$ are doubling (i.e., if $A(2t) \le CA(t)$, and similarly for $\bar{A}$), then $\bar{A} \in B_p$ if and only if

$$\int_c^\infty \left(\frac{t^p}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty.$$

Remark 1. *As we will see with specific examples below, if $\bar{A} \in B_{p'}$, then $\bar{A}(t) \lesssim t^{p'}$ and $A(t) \gtrsim t^p$.*

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Given p, $1 < p < \infty$, let A and B be Young functions such that $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. We say that the pair of weights (u, v) satisfies an A_p bump condition with respect to A and B if

$$\sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{B,Q} < \infty, \tag{1}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^d$, and the Luxemburg norm is defined by

$$\|f\|_{A,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q A\big(|f(x)|/\lambda\big)\,dx \le 1\right\}.$$

If (1) holds, then it is conjectured that

$$T : L^p(v) \to L^p(u). \tag{2}$$

Similarly, if the pair (u, v) satisfies the weaker condition

$$\sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{p',Q} < \infty, \tag{3}$$

then the conjecture is that

$$T : L^p(v) \to L^{p,\infty}(u). \tag{4}$$

The conditions (1) and (3) are referred to as A_p bump conditions because they may be thought of as the classical two-weight A_p condition with the localized L^p and $L^{p'}$ norms "bumped up" in the scale of Orlicz spaces. These conditions have a long history. They first appeared in connection with estimates for integral operators related to the spectral theory of Schrödinger operators: see Fefferman [8] and Chang–Wilson–Wolff [1]. These papers demonstrate a very close connection with uncertainty principles; for this aspect also see the very interesting paper of Pérez and Wheeden [32]. The bump condition considered in [1, 8] was the Fefferman–Phong condition that used "power" bumps: i.e., Young functions of the form $A(t) = t^{rp}$, $r > 1$. Power bumps were independently introduced by Neugebauer [28]. Bump conditions in full generality were introduced by Pérez [29, 30, 31].

The conjectured strong and weak-type inequalities for singular integrals have been studied extensively, but the full results have proved elusive. The strong-type conjecture is true for operators of bounded complexity (e.g., the Hilbert transform, the Riesz transforms and the Buerling–Ahlfors operators): see [2].

Lerner [15] proved that it holds for any Calderón-Zygmund operator if $p > n$. Very recently, it was proved for $p = 2$ in any dimension and for any Calderón–Zygmund operator using Bellman function techniques: see [19].

Theorem 1.1. *Given $p = 2$, suppose the pair of weights (u, v) satisfies (1), where $\bar{A} \in B_2$ and $\bar{B} \in B_2$. Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^2(u)} \le C\|f\|_{L^2(v)}$, where C depends only on T, the dimension d, and the suprema in (1).*

Remark 2. *It turns out that extending this theorem to $p \neq 2$, and especially strengthening it by replacing two-side bump conditions (1) by weaker one-side conditions (7) and (8) is difficult.*

Certain additional results are known in the special case that A and B are "log-bumps": that is, of the form

$$A(t) = t^p \log(e+t)^{p-1+\delta}, \quad \bar{A}(t) \approx \frac{t^{p'}}{\log(e+t)^{1+\delta'}}, \tag{5}$$

$$B(t) = t^{p'} \log(e+t)^{p'-1+\delta}, \qquad \bar{B}(t) \approx \frac{t^{p}}{\log(e+t)^{1+\delta''}}, \tag{6}$$

where $\delta > 0$, $\delta' = \delta/(p-1)$, $\delta'' = \delta/(p'-1)$. But even in this case the result for all Calderón–Zygmund operators was unknown. The weak-type conjecture is only known for log bumps: see [5]. For a complete history of both conjectures and these partial results, we refer the reader to the work of Pérez, Cruz-Uribe, and Martell [2, 3, 4, 6, 7], and Treil, Volberg, and Zheng [38], and the extensive references they contain.

One can motivate the conjectures (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) (and the related conjectures we consider below) by considering a pair of conjectures due to Muckenhoupt and Wheeden. First, they conjectured that a singular integral operator (in particular, the Hilbert transform) satisfies (2) provided that the Hardy-Littlewood maximal operator satisfies

$$M : L^p(v) \to L^p(u),$$
$$M : L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'}).$$

They also conjectured that (4) holds if the maximal operator satisfies the second, $L^{p'}$ inequality. Pérez [31] (see also [3]) proved that a sufficient condition for each of these estimates to hold for M is that the pair (u, v) satisfies

$$\sup_Q \|u^{1/p}\|_{p,Q} \|v^{-1/p}\|_{B,Q} < \infty, \tag{7}$$

$$\sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{p',Q} < \infty; \tag{8}$$

in particular, both these conditions hold if (1) holds.

Though intuitively appealing, both of the Muckenhoupt-Wheeden conjectures are false. A counter-example to the strong-type conjecture was recently found by Reguera and Scurry [34]. The weak-type conjecture is an easy consequence of the two-weight, weak $(1, 1)$ conjecture (also due to Muckenhoupt and Wheeden), but this was recently proved false by Reguera and Thiele [35]. While this does not show the conjecture false, it strongly suggests that it is. And as a byproduct of our approach to our main results we show that the weak-type conjecture is also false; as a consequence we get another proof that their weak $(1, 1)$ conjecture is false.

Given the falsity of the Muckenhoupt-Wheeden conjectures (even for $p = 2$), the A_p bump conjectures become even more interesting. And Theorem 1.1 and many other results listed above strongly suggest that it should hold in the full range of p, dimensions, and Calderón–Zygmund operators. Here we consider two even stronger conjectures, motivated by the fact that the "separated" bump conditions (7) and (8) are sufficient for the maximal operator inequalities that they posited.

Conjecture 1. *Given p, $1 < p < \infty$, suppose the pair of weights (u, v) satisfies (7) and (8), where $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. Then every Calderón-Zygmund singular*

integral operator T satisfies $\|Tf\|_{L^p(u)} \le C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the suprema in (7) *and* (8).

Conjecture 2. *Given p, $1 < p < \infty$, suppose the pair of weights (u, v) satisfies* (8) *where $\bar{A} \in B_{p'}$. Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^{p,\infty}(u)} \le C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the supremum in* (8).

We can prove Conjecture 1 in the special case when A, B are log bumps.

Theorem 1.2. *Given p, $1 < p < \infty$, suppose the pair of weights (u, v) satisfies* (7) *and* (8), *where A and B are log bumps of the form* (5) *and* (6). *Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^p(u)} \le C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the suprema in* (7) *and* (8).

Our techniques also immediately yield Conjecture 2 for log bumps. This gives a new proof of the result originally proved in [5]; for completeness we include it here.

Theorem 1.3. *Given p, $1 < p < \infty$, suppose the pair of weights (u, v) satisfies* (8) *where A is a log bump of the form* (5). *Then every Calderón-Zygmund singular integral operator T satisfies $\|Tf\|_{L^{p,\infty}(u)} \le C\|f\|_{L^p(v)}$, where C depends only on T, the dimension d, and the supremum in* (8).

Remark 3. *Theorems 1.2 and 1.3 are both sharp, in the sense that if we take $\delta = 0$ in the definition of A or B, then there exist pairs of weights that satisfy the bump conditions but such that the corresponding norm inequalities are false. For details, see* [3].

Our proof of Theorems 1.2 and 1.3 depends heavily on the machinery developed to prove the one-weight A_2 conjecture [2, 9, 14]. In turn many of the techniques used to prove the A_2 conjecture have their genesis in nonhomogeneous Harmonic Analysis. In particular, they go back to the random geometric constructions introduced in [21, 22, 23]. For a summary of these results, see [39].

The method of the proof allows us to extend it one step further to loglog-bumps (some of them) and to prove

Theorem 1.4. *Given p, $1 < p < \infty$, suppose the pair of gauge functions A, B satisfies loglog-bump condition* (1) *with* **sufficiently large positive** *δ, and the pair of weights (u, σ) satisfies* (37). *Given any dyadic shift S of complexity (m, n), $\tau = \max(m, n)+1$, $\|S(f\sigma)\|_{L^p(u)} \le C\tau^2\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the suprema in* (37).

Theorem 1.5. *Given p, $1 < p < \infty$, suppose that the gauge function A satisfies loglog-bump condition* (9) *with* **sufficiently large positive** *δ, and weights (u, σ) satisfies condition* (37). *Given any dyadic shift S of complexity (m, n), $\tau = \max(m, n) + 1$, $\|S(f\sigma)\|_{L^{p,\infty}(u)} \le C\tau^2\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the supremum in* (37).

The reader will see the dictionary between the languages of (u, v) and $u, \sigma)$ in the next Section.In particular, the condition (1) in terms of (u, v) becomes exactly (37) in terms of the pair (u, σ).

The remainder of this paper is organized as follows. In Section 2 we reformulate our results and reduce the problem to proving the corresponding results for a general class of dyadic shift operators (Theorem 2.3 and 2.4). In Section 3 we prove Theorem 2.3. It is important to note that in most of the proof we only need to assume that $\bar{A} \in B_{p'}, \bar{B} \in B_p$; only at one step are we forced to assume that A, B are log bumps. In Section 4 we describe the (minor) changes required to prove Theorem 2.4. In Section 5 we show that the Muckenhoupt-Wheeden conjecture for the weak-type inequality is false. Finally, in Section 6, we prove the loglog-bump theorems.

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2. Preliminary Results

Hereafter, we will use the notation

$$\langle f \rangle_Q = \frac{1}{|Q|} \int\limits_Q f(x)dx.$$

We also restate our weighted norm inequalities in an equivalent form. Let $\sigma = v^{1-p'}$; then we can rewrite (7) and (8) as

$$\sup_Q \langle u \rangle_Q^{1/p} \|\sigma^{1/p'}\|_{B,Q} < \infty, \tag{9}$$

$$\sup_Q \|u^{1/p}\|_{A,Q} \langle \sigma \rangle_Q^{1/p'} < \infty. \tag{10}$$

By the properties of the Luxemburg norm we have that either condition implies the two-weight A_p condition:

$$\sup_Q \langle u \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'} < \infty. \tag{11}$$

Similarly, we can restate the conclusions of Theorems 1.2 and 1.3 as

$$\|T(f\sigma)\|_{L^p(u)} \leq C\|f\|_{L^p(\sigma)}, \qquad \|T(f\sigma)\|_{L^{p,\infty}(u)} \leq C\|f\|_{L^p(\sigma)}.$$

The B_p condition is closely connected to a generalization of the maximal operator. Recall that the Hardy-Littlewood maximal operator is defined to be

$$Mf(x) = \sup_{Q \ni x} \langle |f| \rangle_Q = \sup_{Q \ni x} \|f\|_{1,Q}.$$

Given a Young function A, we define the Orlicz maximal operator M_A by

$$M_A f(x) := \sup_{Q \ni x} \|f\|_{A,Q}.$$

The following result is due to Pérez [31] (see also [3]).

Theorem 2.1. *Fix p, $1 < p < \infty$, and let A be a Young function such that $A \in B_p$. Then $M_A : L^p \to L^p$.*

The B_p condition is also sufficient for a two-weight norm inequality for the Hardy-Littlewood maximal operator. This result is also due to Pérez [31, 3].

Theorem 2.2. *Fix p, $1 < p < \infty$, and let B be a Young function such that $\bar{B} \in B_p$. If the pair of weights (u, σ) satisfies*

$$\sup_Q \langle u \rangle_Q^{1/p} \|\sigma^{1/p'}\|_{B,Q} < \infty, \tag{12}$$

then

$$\|M(f\sigma)\|_{L^p(u)} \leq C\|f\|_{L^p(\sigma)}. \tag{13}$$

Remark 4. *The bump condition* (12) *is necessary in the following sense: suppose that B is a function such that whenever* (12) *holds, the maximal operator satisfies* (13)*. Then $\bar{B} \in B_p$. See* [31].

We now turn to the definition of the dyadic Haar shift operators that will replace an arbitrary Calderón-Zygmund operator.

Definition 1. *Given a dyadic cube Q, h_Q is a (generalized) Haar function associated to a cube Q if*

$$h_Q(x) = \sum_{Q' \in ch(Q)} c_{Q'} \chi_{Q'}(x),$$

where $ch(Q)$ is the set of dyadic children of Q and $|c_{Q'}| \leq 1$.

Definition 2. *We say that an operator S has a Haar shift kernel of complexity (m, n) if*

$$Sf(x) = \sum_Q S_Q(f),$$

where

$$S_Q(f) = \frac{1}{|Q|} \sum_{\substack{Q', Q'' \subset Q \\ \ell(Q') = 2^{-n}\ell(Q) \\ \ell(Q'') = 2^{-m}\ell(Q)}} (f, h_{Q'}) h_{Q''}$$

and $h_{Q'}$ and $h_{Q''}$ are generalized Haar functions associated to the cubes Q' and Q'' respectively. We say that S is a Haar shift of complexity (m, n) if it has a Haar shift kernel of complexity (m, n), and it is bounded on $L^2(dx)$.

By the decomposition theorem of Hytönen [9, 10], to prove Theorems 1.2 and 1.3 it will suffice to prove that they hold for Haar shift operators of complexity (m, n) with a constant that grows polynomially in $\tau = \max(m, n) + 1$. More precisely we will prove the following.

Theorem 2.3. *Given p, $1 < p < \infty$, suppose the pair of gauge functions A, B satisfies log-bump conditions 5, 6 and the pair of weights (u, v) satisfies* (9) *and* (10)*. Given any dyadic shift S of complexity (m, n), $\tau = \max(m, n) + 1$, $\|S(f\sigma)\|_{L^p(u)} \leq C\tau^2\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the suprema in* (9) *and* (10).

Theorem 2.4. *Given p, $1 < p < \infty$, suppose that the gauge function A satisfies log-bump condition 5 and weights (u, v) satisfies condition* (10)*. Given any dyadic shift S of complexity (m, n), $\|S(f\sigma)\|_{L^{p,\infty}(u)} \leq C\tau^2\|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the supremum in* (10).

3. Proof of Theorem 2.3

To prove the strong-type inequality we follow the argument used by Hytönen and Lacey [11] in the one-weight case, which in turn refines the proof given in [2]. Fix a function f that is bounded and has compact support. For each $N > 0$, let $Q_N = [-2^N, 2^N]^d$. By Fatou's lemma,

$$\|S(f\sigma)\|_{L^p(u)} \leq \liminf_{N\to\infty} \left(\int_{Q_N} |S(f\sigma)(x) - m_{S(f\sigma)}|^p u(x)\,dx \right)^{1/p},$$

where $m_{S(f\sigma)}$ is the median value of $S(f\sigma)$ on Q_N. Fix N. Using the remarkable decomposition theorem of Lerner [15], they show that there exists a family of dyadic cubes $\mathcal{L} = \{Q_j^k\}$ and pairwise disjoint sets $\{E_j^k\}$ such that $E_j^k \subset Q_j^k$, $|E_j^k| \geq \frac{1}{2}|Q_j^k|$, and

$$\left(\int_{Q_N} |S(f\sigma)(x) - m_{S(f\sigma)}|^p u(x)\,dx \right)^{1/p} \leq C\tau\|M(f\sigma)\|_{L^p(u)} + C\,\tau \sum_{i=1}^{\tau} \| \sum_{j,k} \langle |f|\sigma\rangle_{(Q_j^k)^i} \chi_{Q_j^k} \|_{L^p(u)}. \tag{14}$$

(Here, given a dyadic cube Q, Q^i denotes the i-th parent of Q.) The linear dependence on τ in (14) can be found in [11]: see Lemma 2.4 and the discussion following it. Alternatively, we can deduce it from the argument in [2] if: 1) we combine it with the unweighted weak-type estimate with the right dependence on τ in [9, 14]; 2) we precede the weak-type estimate of $1_Q S(1_{Q^\tau} f)$ by a careful pointwise estimate of this function on Q. This lets us reduce the weak-type estimate of this expression to the weak-type estimate of $1_Q S(1_Q f)$ (with an error term that can be controlled).

By Theorem 2.2, $\|M(f\sigma)\|_{L^p(u)} \leq C\|f\|_{L^p(\sigma)}$. Therefore, it remains to estimate the second term in (14). Again, following [11], we show that this reduces to a two-weight estimate for a positive Haar shift operator.

We reorder the sum as follows: fix an integer $i \in [1, \tau]$ and sum over every cube $Q = (Q_j^k)^i$ and then over all cubes $Q_r^s \in \mathcal{L}$ such that $(Q_r^s)^i = Q$. Then we have that

$$\sum_{j,k} \langle |f|\sigma\rangle_{(Q_j^k)^i} \chi_{Q_j^k} = \sum_Q \langle |f|\sigma\rangle_Q \sum_{\substack{R\in\mathcal{L}\\ R^i=Q}} \chi_R = \sum_Q \chi_Q^i \langle |f|\sigma\rangle_Q = S_{\mathcal{L}}^i(|f|\sigma),$$

where the last sum is taken over all dyadic cubes Q and

$$\chi_Q^i = \sum_{\substack{R\in\mathcal{L}\\ R^i=Q}} \chi_R.$$

Clearly, $S_{\mathcal{L}}$ (hereafter we omit the superscript i) is a positive operator. We claim that it is in fact a positive Haar shift of complexity at most $(0, \tau - 1)$. From the definition we have that

$$S_{\mathcal{L}} f = \sum_{Q\in\mathcal{L}} \frac{1}{|Q|} \chi_Q^i \int_Q f,$$

and so in the notation used above we have that

$$S_Q = \frac{1}{|Q|} \chi_Q^i \int_Q f,$$

$Q' = Q$, $h_{Q'} = \chi_Q$, the Q'' are all the $(i-1)$-children of Q, and $h_{Q''} = \sum_{R \in ch(Q'')} c_R \chi_R$, where $c_R = 1$ if $R \in \mathcal{L}$ and $c_R = 0$ otherwise. Thus $S_\mathcal{L}$ has a Haar shift kernel of complexity $(0, i-1)$, $i \leq \tau$. To see that it is bounded on L^2, we use the properties of the cubes Q_j^k. By duality, there exists $g \in L^2$, $\|g\|_2 = 1$, such that

$$\|S_\mathcal{L} f\|_2^2 = \int_{\mathbb{R}^d} \sum_{j,k} \langle f \rangle_{(Q_j^k)^i} \chi_{Q_j^k}(x) g(x)\, dx$$
$$\leq 2 \sum_{j,k} \langle f \rangle_{(Q_j^k)^i} \langle g \rangle_{Q_j^k} |E_j^k| \leq 2 \int_{\mathbb{R}^d} Mf(x) Mg(x)\, dx.$$

The last integral is bounded by $\|f\|_2 \|g\|_2$ by Hölder's inequality and the unweighted L^2 inequality for the maximal operator.

Remark 5. *It follows from this argument that both $S_\mathcal{L}$ and its adjoint $S_\mathcal{L}^*$ are positive Haar shifts with uniform bounds.*

Definition. Given a positive Haar shift operator S, define the associated maximal singular integral operator by

$$S_\sharp(x) := \sup_{0<\epsilon\leq v<\infty} S_{\epsilon,v}(x) = \sup_{0<\epsilon\leq v<\infty} \sum_{Q \in \mathcal{D},\, \epsilon \leq \ell(Q) \leq v} S_Q f(x)\,.$$

To prove that $\|S_\mathcal{L}(f\sigma)\|_{L^p(u)} \leq C\tau \|f\|_{L^p(\sigma)}$, we use the following result that is essentially due to Sawyer and can found in Hytönen and Lacey [11] and Hytönen, *et al.* [12]. The precise statement below is gotten by combining Theorem 4.7 of [12] with Corollary 3.2 and Lemma 3.3 of [11].

Theorem 3.1. *Let S be a positive Haar shift of complexity (m, n). Then the associated maximal singular integral $S_\sharp$ satisfies*

$$(15) \quad \|S_\sharp(\cdot\sigma)\|_{L^p(\sigma)\to L^p(u)} \leqslant \tau \|M(\cdot\sigma)\|_{L^p(\sigma)\to L^p(u)} + \sup_Q \frac{\|\chi_Q S(\chi_Q \sigma)\|_{L^p(u)}}{\sigma(Q)^{\frac{1}{p}}} + \sup_Q \frac{\|\chi_Q S^*(\chi_Q u)\|_{L^p(\sigma)}}{u(Q)^{\frac{1}{p}}}.$$

Remark 6. *Testing conditions of this kind were first proved by E. Sawyer* [37] *for positive operators. Later, this result was proved by Nazarov, Treil and Volberg* [20, 24] *for all well localized operators (in particular, all Haar shifts) when $p = 2$. Also see Theorem 5.1 below. It is not known if Theorem 3.1 is true for all Haar shifts when $p \neq 2$ even if $S_\sharp$ is replaced by S.*

We will now use Theorem 3.1 to show that if S is any positive Haar shift operator, then the right-hand side of (15) is finite if our bump conditions are satisfied. As before, by Theorem 2.2 the first term is bounded by $C\tau$. It remains to estimate the two testing conditions. We will estimate the first; the estimate for S^* is gotten in essentially the same fashion. (See Remark 7 below.)

Fix a cube Q_0; using the notation from the definition of a Haar shift, we have that

$$(16) \quad \chi_{Q_0} S(\chi_{Q_0}\sigma) = \sum_{R \subset Q_0} S_R(\sigma) + \chi_{Q_0} \sum_{R,\, Q_0 \subset R} S_R(\chi_{Q_0}\sigma) \leqslant \sum_{R \subset Q_0} S_R(\sigma) + \chi_{Q_0} \langle \sigma \rangle_{Q_0}.$$

The second inequality is straightforward: see, for instance, [9, 11, 12, 14]. As we noted above, the pair (u, σ) satisfies the two-weight A_p condition (11). Therefore, the $L^p(u)$ norm of the second term is bounded by

$$\|\chi_{Q_0}\|_{L^p(u)} \langle\sigma\rangle_{Q_0} = \langle u\rangle_{Q_0}^{1/p} \langle\sigma\rangle_{Q_0}^{1/p'} \sigma(Q_0)^{1/p} \leq C\sigma(Q_0)^{1/p}\,.$$

To estimate the $L^p(u)$ norm of the first term, we form the following decomposition (see [11]):

$$\begin{aligned}
\mathcal{K} = \mathcal{K}_i &= \{Q \subset Q_0 \colon \ell(I) = 2^{i+\tau n}\},\ n \in \mathbb{Z}_+;\\
\mathcal{K}_a &= \{Q \in \mathcal{K} \colon 2^a \leqslant \langle u\rangle_Q^{\frac{1}{p}} \langle\sigma\rangle_Q^{\frac{1}{p'}} < 2^{a+1}\};\\
\mathcal{P}_0^a &= \text{all maximal cubes in } \mathcal{K}_a;\\
\mathcal{P}_n^a &= \left\{\text{maximal cubes } P' \subset P \in \mathcal{P}_{n-1}^a, \text{ such that } \langle\sigma\rangle_{P'} > 2\langle\sigma\rangle_P\right\};\\
\mathcal{P}^a &= \bigcup_{n\geqslant 0} \mathcal{P}_n^a.
\end{aligned}$$

Hereafter we suppress the index i; this will give us a sum with $\tau + 1$ terms. Given $Q \in \mathcal{K}_a$, let $\Pi(Q)$ denote the minimal principal cube that contains it, and define

$$\mathcal{K}_a(P) = \{Q \in \mathcal{K}_a : \Pi(Q) = P\}.$$

We will estimate the $L^p(u)$ norm of the first sum on the right-hand side of (16) using the exponential decay distributional inequality originated in [16]. (This inequality was subsequently improved in the sense of establishing the right dependence on the complexity of the shift in [14, 11].) Below, S is any positive generalized Haar shift that is bounded on unweighted L^2. In particular, we will take S to be one of the positive Haar shifts $S_{\mathcal{L}}$ from above.

Theorem 3.2. *There exists a constant c, depending only on the dimension and the unweighted L^2 norm of the shift, such that for any $P \in \mathcal{P}^a$,*

$$u\left(x \in P :\ |S_{\mathcal{K}_a(P)}(\sigma)| > t\frac{\sigma(P)}{|P|}\right) \lesssim e^{-ct}\, u(P).$$

It follows from Theorem 3.2 that for some positive constant c,

$$\|\sum_{R\subsetneq Q} S_R(\sigma)\|_{L^p(u)} \leqslant C\tau \sum_a \left(\sum_{P\in\mathcal{P}^a} u(P)\left(\frac{\sigma(P)}{|P|}\right)^p\right)^{\frac{1}{p}}. \tag{17}$$

We sketch the proof of (17) following the beautiful calculations in [11]:

$$\sum_{R\subsetneq Q} S_R(\sigma) = \sum_{i=0}^{\tau} \sum_a \sum_{P\in\mathcal{P}^a} S_{\mathcal{K}^a(P)}(\sigma)\,,$$

and so

$$\|\sum_{R\subsetneq Q} S_R(\sigma)\|_{L^p(u)} \leqslant (\tau+1) \sum_a \|\sum_{P\in\mathcal{P}^a} S_{\mathcal{K}^a(P)}(\sigma)\|_{L^p(u)}\,.$$

Fix a. Using Fubini's theorem we write

$$
\begin{aligned}
&\|\sum_{P\in\mathcal{P}^a} S_{\mathcal{K}^a(P)}(\sigma)\|_{L^p(u)} \\
&\qquad = \left(\int \left(\sum_j \sum_{P\in\mathcal{P}^a} \chi_{\{S_{\mathcal{K}^a(P)}(\sigma)\in(j,j+1)\frac{v(P)}{|P|}\}} S_{\mathcal{K}^a(P)}(\sigma)(x)\right)^p u(x)\,dx\right)^{1/p} \\
&\qquad \leq \sum_j (j+1)\left(\int \left[\sum_{P\in\mathcal{P}^a} \chi_{\{S_{\mathcal{K}^a(P)}(\sigma)\in(j,j+1)\frac{\sigma(P)}{|P|}\}} \frac{\sigma(P)}{|P|}\right]^p u(x)\,dx\right)^{1/p}.
\end{aligned}
$$

By the choice of the stopping cubes $P\in\mathcal{P}^a$ we have that

$$
\left[\sum_{P\in\mathcal{P}^a} \chi_{\{S_{\mathcal{K}^a(P)}(\sigma)\in(j,j+1)\frac{\sigma(P)}{|P|}\}} \frac{\sigma(P)}{|P|}\right]^p \lesssim \sum_{P\in\mathcal{P}^a} \chi_{\{S_{\mathcal{K}^a(P)}(\sigma)\in(j,j+1)\frac{\sigma(P)}{|P|}\}} \left(\frac{\sigma(P)}{|P|}\right)^p.
$$

This follows because the ratios $\frac{\sigma(P)}{|P|}$ in the sum on the left are super-exponential. This beautiful observation from [11] lets us write

$$
\|\sum_{P\in\mathcal{P}^a} S_{\mathcal{K}^a(P)}(\sigma)\|_{L^p(u)} \lesssim \sum_j (j+1)\left(\sum_{P\in\mathcal{P}^a} \left(\frac{\sigma(P)}{|P|}\right)^p u(S_{\mathcal{K}^a(P)}(\sigma)\in(j,j+1)\frac{\sigma(P)}{|P|})\right)^{1/p}.
$$

Then by the distributional inequality from Theorem 3.2:

$$
\|\sum_{P\in\mathcal{P}^a} S_{\mathcal{K}^a(P)}(\sigma)\|_{L^p(u)} \lesssim \sum_j (j+1)e^{-cj/p}\left(\sum_{P\in\mathcal{P}^a} \left(\frac{\sigma(P)}{|P|}\right)^p u(P)\right)^{1/p}.
$$

This gives us (17).

It is at this point in the proof that we can no longer assume that our pair of weights (u,v) satisfies the general A_p bump condition and we must instead make the more restrictive assumption that we have log bumps. Before doing so, however, we want to show how the proof goes and where the problem arises for general bumps. We will then give the modification necessary to make this argument work for log bumps.

Define the sequence

$$
\mu_Q = \begin{cases} |P|, & Q=P\,\text{, for some cube } P\in\mathcal{P}^a \\ 0, & \text{otherwise;} \end{cases}
$$

then the inner sum in (17) becomes

$$
\sum_{Q\subset Q_0} \frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \mu_Q.
$$

But by Hölder's inequality in the scale of Orlicz spaces,

$$
\frac{\sigma(Q)}{|Q|} = \langle \sigma^{\frac{1}{p}}\sigma^{\frac{1}{p'}}\rangle_Q \leqslant C\|\sigma^{\frac{1}{p'}}\|_{Q,B}\|\sigma^{\frac{1}{p}}\|_{Q,\bar{B}} \leqslant \|\sigma^{\frac{1}{p'}}\|_{Q,B} \inf_{x\in Q} M_{\bar{B}}(\sigma^{\frac{1}{p}}\chi_Q). \tag{18}
$$

Therefore, by (9),

$$
\sum_{Q\subset Q_0} \frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \mu_Q \leqslant K^p \sum_{Q\subset Q_0} \mu_Q \inf_{x\in Q} M_{\bar{B}}(\sigma^{\frac{1}{p}}\chi_Q)^p. \tag{19}
$$

To complete the proof we need two lemmas. The first can be found in [16].

Lemma 3.3. *$\{\mu_Q\}$ is a Carleson sequence.*

The second is a folk theorem; a proof can be found in [17].

Lemma 3.4. *If $\{\mu_Q\}$ is a Carleson sequence, then*

$$\sum_{Q\subset Q_0} \mu_Q \inf_Q \chi_{Q_0} F(x) \lesssim \int_{Q_0} F(x)dx.$$

Combining these two lemmas with Theorem 2.1 (since $\bar{B} \in B_p$) we see that

$$\begin{aligned}\sum_Q \frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \mu_Q &\leqslant K^p \sum_{Q\,,\,Q\subset Q_0} \mu_Q \inf_{x\in Q} M_{\bar{B}}(\sigma^{\frac{1}{p}}\chi_{Q_0})^p \\ &\lesssim K^p \|M_{\bar{B}}(\sigma^{\frac{1}{p}}\chi_{Q_0})\|^p_{L^p(dx)} \lesssim K^p\|\sigma^{\frac{1}{p}}\chi_{Q_0}\|^p_{L^p(dx)} = K^p\sigma(Q_0).\end{aligned}$$

This would complete the proof except that we must now sum over a, and in (17) this sum goes from $-\infty$ to the logarithm of the two-weight A_p constant of the pair (u, σ). We cannot evaluate this sum unless we can modify the above argument to yield a decay constant in a. In the one-weight argument in [11] the authors could use that parameter a above run from 0 to the logarithm of A_p characteristic, this is just by use of an obvious property of A_p weights saying that A_p characteristic of any weight is at least 1; in our case of two weights the joint A_p characteristic can be as small as it wishes to be, and we have to sum up along infinitely many parameters a. We are able to get such a decay constant only by assuming that we are working with log bumps.

We modify the above argument as follows. Essentially, we will use the properties of log bumps to replace $\bar{B}$ with a slightly larger Young function. Define $B_0(t) = t^{p'}\log(e+t)^{p'-1+\frac{\delta}{2}}$; then we again have that $\bar{B}_0 \in B_p$. Instead of (19) we will prove that there exists γ, $0 < \gamma < 1$, such that

$$\sum_{Q\subset Q_0} \frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \mu_Q \leqslant K^{(1-\gamma)p}2^{a\gamma p} \sum_{Q\subset Q_0} \mu_Q \inf_{x\in Q} M_{\bar{B}_0}(\sigma^{\frac{1}{p}}\chi_{Q_0})^p. \tag{20}$$

Given inequality (20), we can repeat the argument above, but we now have the decay term $2^{a\gamma p}$ which allows us to sum in a and get the desired estimate.

To prove (20) suppose for the moment that there exists γ such that

$$\|\sigma^{\frac{1}{p'}}\|_{Q,B_0} \leq C_1\|\sigma^{\frac{1}{p'}}\|^{1-\gamma}_{Q,B}\|\sigma^{\frac{1}{p'}}\|^{\gamma}_{L^{p'}(Q,dx/|Q|)}\,. \tag{21}$$

Given this, fix a cube $Q \in \mathcal{P}^a$—we can do this since otherwise $\mu_Q = 0$. Then

$$\begin{aligned}
&\frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \\
&\qquad \leq \langle u\rangle_Q \|\sigma^{1/p'}\|_{B_0,Q}^p \|\sigma^{1/p}\|_{\bar{B}_0,Q}^p \\
&\qquad \leq \langle u\rangle_Q \|\sigma^{1/p'}\|_{B,Q}^{(1-\gamma)p} \|\sigma^{1/p'}\|_{L^{p'}(Q,dx/|Q|)}^{\gamma p} \|\sigma^{1/p}\|_{\bar{B}_0,Q}^p \\
&\qquad = (\langle u\rangle_Q^{1/p} \|\sigma^{1/p'}\|_{B,Q})^{(1-\gamma)p} \cdot (\langle u\rangle_Q^{1/p} \|\sigma^{1/p'}\|_{L^{p'}(Q,dx/|Q|)})^{\gamma p} \cdot \|\sigma^{1/p}\|_{\bar{B}_0,Q}^p \\
&\qquad \leq K^{(1-\gamma)p} \cdot (\langle u\rangle_Q^{1/p} \langle \sigma\rangle_Q^{1/p'})^{\gamma p} \cdot \|\sigma^{1/p}\|_{\bar{B}_0,Q}^p \\
&\qquad \leq K^{(1-\gamma)p} \cdot 2^{a\gamma p} \cdot \|\sigma^{1/p}\|_{\bar{B}_0,Q}^p \\
&\qquad \leq K^{(1-\gamma)p} \cdot 2^{a\gamma p} \cdot \inf_{x\in Q} M_{\bar{B}_0}(\sigma^{\frac{1}{p}}\chi_{Q_0})^p.
\end{aligned}$$

Inequality (20) now follows immediately.

Therefore, to complete the proof we must establish (21). By the rescaling properties of the Luxemburg norm [3, Section 5.1], the right-hand side of this inequality is equal to

$$\|\sigma^{\frac{1-\gamma}{p'}}\|_{C,Q} \|\sigma^{\frac{\gamma}{p'}}\|_{p'/\gamma,Q},$$

where $C(t) = B(t^{\frac{1}{1-\gamma}})$. Therefore, by the generalized Hölder's inequality in Orlicz spaces ([3, Lemma 5.2]), inequality (21) holds if for all $t > 1$,

$$C^{-1}(t) t^{\frac{\gamma}{p'}} \lesssim B_0^{-1}(t). \tag{22}$$

A straightforward calculation (see [3, Section 5.4]) shows that

$$C^{-1}(t) = B^{-1}(t)^{1-\gamma} \approx \frac{t^{\frac{1-\gamma}{p'}}}{\log(e+t)^{\frac{1-\gamma}{p}+\frac{\delta(1-\gamma)}{p'}}}, \qquad B_0^{-1}(t) \approx \frac{t^{\frac{1}{p'}}}{\log(e+t)^{\frac{1}{p}+\frac{\delta}{2p'}}}.$$

By equating the exponents on the logarithm terms, we see that (22) holds if we take

$$\gamma = \frac{\delta}{2(p'-1+\delta)}.$$

Therefore, with this value of γ inequality (21) holds, and this completes our proof.

For the convenience of the reader we also give below a direct proof of (21), which obviously follows from this lemma:

Lemma 3.5. *Le μ be a probability measure, f be a non-negative measurable function. Let B, B_0 be logarithmic bumps as in (6) with $\delta = \tau$ and $\delta = \frac{\tau}{2}$ correspondingly. Then there exists an absolute constant C and $\gamma = \gamma(p', \tau) > 0$ such that*

$$\|f\|_{B_0,\mu} \leq C\, \|f\|_{B,\mu}^{1-\gamma}\, \|f\|_{L^{p'}(\mu)}^{\gamma}\,. \tag{23}$$

Proof. We will see now that $\gamma = \frac{1}{2+(p'-1)\frac{2}{\tau}}$. Denote $\Delta := \int |f|^{p'}\, d\mu$. We first consider the case when

$$\|f\|_{B,\mu} = 1\,. \tag{24}$$

Of course we can think that Δ is sufficiently small, otherwise (23) can be achieved by choosing large C. Choose $\epsilon < 1$ and K (in this order) later. Write

$$\int \frac{f^{p'}}{\epsilon^{p'}} \log^{p'-1+\frac{\tau}{2}}(e+\frac{f}{\epsilon})\, d\mu \le \int_{f\le K\epsilon} \cdots + \int_{f\ge K\epsilon} \cdots \le \frac{\Delta}{\epsilon^{p'}}[\log(e+K)]^{p'-1+\frac{\tau}{2}} +$$

$$\int_{f\ge K\epsilon} \frac{f^{p'}}{\epsilon^{p'}} \frac{\log^{p'-1+\tau}(e+\frac{f}{\epsilon})}{[\log(e+K)]^{\frac{\tau}{2}}}\, d\mu \le \frac{\Delta}{\epsilon^{p'}}[\log(e+K)]^{p'-1+\frac{\tau}{2}} + \int \frac{f^{p'}}{\epsilon^{p'}} \frac{\log^{p'-1+\tau}(\frac{e}{\epsilon}+\frac{f}{\epsilon})}{[\log(e+K)]^{\frac{\tau}{2}}}\, d\mu \le$$

$$\frac{\Delta}{\epsilon^{p'}}[\log(e+K)]^{p'-1+\frac{\tau}{2}} + \frac{1}{\epsilon^{p'}[\log(e+K)]^{\tau/2}}\left[\int f^{p'} \log^{p'-1+\tau}(e+f)\, d\mu + \int f^{p'} (\log\frac{1}{\epsilon})^{p'-1+\tau}\right]$$

$$\le \frac{\Delta}{\epsilon^{p'}}[\log(e+K)]^{p'-1+\frac{\tau}{2}} + \frac{1}{\epsilon^{p'}[\log(e+K)]^{\tau/2}}\left[1+\Delta(\log\frac{1}{\epsilon})^{p'-1+\tau}\right].$$

In the last line we used (24). Now choose

$$[\log(e+K)]^{\tau/2} \approx \frac{1}{\epsilon^{p'}}\,,\ \text{then}\ [\log(e+K)]^{p'-1+\tau/2} \approx (\frac{1}{\epsilon^{p'}})^{1+(p'-1)\frac{2}{\tau}} =: (\frac{1}{\epsilon^{p'}})^{c}\,.$$

Here $c = 1+(p'-1)\frac{2}{\tau}$. Choose ϵ from

$$\Delta = (\epsilon^{p'})^{1+c}\,.$$

That is

$$\epsilon = (\Delta^{1/p'})^{\gamma} = \|f\|^{\gamma}_{L^{p'}(\mu)}\,,\ \gamma = \frac{1}{1+c}\,.$$

Our long calculation now shows the right hand side is $\asymp C_0$, this means that Luxemburg's norm

$$\|f\|_{B_0,\mu} \le C\,\epsilon = C\,\|f\|^{\gamma}_{L^{p'}(\mu)}\,.$$

Now we can get rid of (24), and using linearity of Luxemburg's norms, we obtain (23). Lemma is proved.

□

Remark 7. *The conjugate testing condition can be verified similarly. By Remark 5 the adjoint S^* is also a Haar shift, and so we can apply the distribution inequality from Theorem 17 to it. Also, the second sum in* (16) *will have the same pointwise estimate (exchanging σ and v) if we replace S with S^*.*

Remark 8. *In the proof of the first testing condition we only used the bump condition* (9)*; to prove the second testing condition we use the second bump condition* (10).

4. Proof of Theorem 2.4

The proof of the weak-type inequality uses essentially the same argument as above; here we sketch the changes required. We repeat the argument that yields inequality (14), replacing the $L^p(u)$ norm with the $L^{p,\infty}(u)$ norm. Since the pair (u, v) satisfies the two-weight A_p condition we have the well known inequality that

$$\|M(f\sigma)\|_{L^{p,\infty}(u)} \le C\|f\|_{L^p(\sigma)},$$

where the constant C depends only on the A_p constant and the dimension. Therefore it remains to estimate the $L^{p,\infty}(u)$ norm of $S_{\mathcal{L}}(|f|\sigma)$. However, from Hytönen, *et al.* [12, Theorem 4.3] we have the following analog of Theorem 3.1.

Theorem 4.1. *Let S be a positive Haar shift of complexity (m,n). Then*

$$\|S(\cdot\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(u)} \leqslant \tau\|M(\cdot\sigma)\|_{L^p(\sigma)\to L^{p,\infty}(u)} + \sup_Q \frac{\|\chi_Q S^*(\chi_Q u)\|_{L^p(\sigma)}}{u(Q)^{\frac{1}{p}}}.$$

Given Theorem 4.1 the argument now proceeds exactly as before, using the bump condition (10) to bound the testing condition. This completes the proof.

5. Counterexamples to Muckenhoupt–Wheeden conjectures

In this section we prove that the weak-type conjecture of Muckenhoupt and Wheeden discussed in the Introduction is false for the Hilbert transform when $p=2$. We in fact prove a stronger result.

For brevity, we introduce some additional notation. Let $\sigma = v^{-1}$ and let $M_\sigma f = M(f\sigma)$. Define M_u, H_σ and H_u similarly, where H is the Hilbert transform. Then we can reformulate the conjecture as follows: if

$$M_u : L^2(u) \to L^2(\sigma). \tag{25}$$

then

$$H_\sigma : L^2(\sigma) \to L^{2,\infty}(u). \tag{26}$$

We will show by contradiction that this is not true in general. Suppose to the contrary that if the pair (u,v) satisfies (25), then (26) holds. Then for any $f\in L^2(u)$ and any cube Q,

$$\int_Q H_\sigma f u\,dx \le \|H_\sigma f\|_{L^{2,\infty}(u)}\|\chi_Q\|_{L^{2,1}(u)} \le \|H_\sigma\|_{L^2(\sigma)\to L^{2,\infty}(u)}\|f\|_{L^2(\sigma)}u(Q)^{1/2}.$$

Let $f = H_u(\chi_Q)$. Then by duality (since H_σ is the adjoint of H_u) we have that the pair (u,v) satisfies the testing condition

$$\int_Q |H_u(\chi_Q)|^2\sigma\,dx \le Cu(Q). \tag{27}$$

The same argument shows that if the pair (u,σ) satisfies

$$M_\sigma : L^2(\sigma) \to L^2(u), \tag{28}$$

then this pair also satisfies the testing condition

$$\int_Q |H_\sigma(\chi_Q)|^2 u\,dx \le C\sigma(Q). \tag{29}$$

However, we have the following testing condition result for the Hilbert transform when $p=2$. This was proved in [39, Chapter 22] (see also [25]).

Theorem 5.1. *Let H be the Hilbert transform. Then*

$$\|H(\cdot\sigma)\|_{L^2(\sigma)\to L^2(u)} \leqslant \|M(\cdot\sigma)\|_{L^2(\sigma)\to L^2(u)} + \|M(\cdot u)\|_{L^2(u)\to L^2(\sigma)} + \sup_Q \frac{\|H(\chi_Q\sigma)\|_{L^2(u)}}{\sigma(Q)^{\frac{1}{2}}} + \sup_Q \frac{\|H(\chi_Q u)\|_{L^2(\sigma)}}{u(Q)^{\frac{1}{2}}}.$$

Remark 9. *We note that the proofs in [39] and [25] can be adapted to prove Theorem 5.1 for an arbitrary operator with Calderón–Zygmund kernel. This was essentially done in [33]. This paper is concerned with one-weight inequalities, but the same argument works with no change in the two weight case.*

Therefore, by our assumption and Theorem 5.1, we have that if a pair of weights (u, σ) satisfies (25) and (28), then $H : L^2(\sigma) \to L^2(u)$. However, this contradicts the counterexample constructed by Reguera and Scurry [34] (in fact, Reguera–Scurry's paper directly disproves the test condition for the Hilbert transform). Therefore, the weak-type conjecture of Muckenhoupt and Wheeden cannot hold.

In fact, we have proved a stronger result.

Theorem 5.2. *There exists a pair of weights* (u, σ) *such that* (25) *and* (28) *hold, but the Hilbert transform does not satisfy the weak-type inequality* $H_\sigma : L^2(\sigma) \to L^{2,\infty}(u)$.

We conclude this section with three remarks. First, Theorem 5.1 is essentially Theorem 22.3 in [39], but it is formulated there in slightly different language. For the convenience of the reader we want to explain why these two results are in fact equivalent.

The main part of Theorem 22.3 in [39] says that if $M_u, M_\sigma, H_u, H_\sigma$ all satisfy the testing conditions (27) and (29) (replacing H with M when dealing with the maximal operator), then all four of them are bounded in corresponding pairs of weighted spaces. This gives us Theorem 5.1. Conversely, suppose that the right-hand side of the inequality in Theorem 5.1 is finite. Then (27) and (29) are both satisfied. Moreover, M_u and M_σ are both bounded on the corresponding pairs of weighted spaces. Therefore, trivially, M_u, M_σ also satisfy the corresponding testing conditions. Thus, all four operators satisfy the testing conditions, and then Theorem 22.3 gives that $H_u : L^2(u) \to L^2(\sigma)$ and $H_\sigma : L^2(\sigma) \to L^2(u)$ as well as the corresponding norm inequalities for the maximal operators. (The latter follows from Sawyer's testing criterion for the two weight boundedness of maximal function [36].)

In the Introduction we noted that the weak-type conjecture we just disproved followed from another conjecture of Muckenhoupt and Wheeden: that

$$u(\{x : |Hf(x)| > t\}) \le \frac{C}{t} \int |f|\, Mu\, dx\,. \tag{30}$$

(This implication is a straightforward duality argument: see [3].) Inequality (30) was disproved by Reguera and Thiele [35]; our result above gives another (indirect) proof of this fact.

Finally, we note that there is a weaker conjecture than (30) which has also been shown to be false. In the one-weight case it was conjectured that for all $w \in A_1$,

$$w(\{x : Hf(x) > t\}) \le \frac{C}{t} [w]_{A_1} \int |f|\, w\, dx\,, \tag{31}$$

where $[w]_{A_1} = \|\frac{Mw}{w}\|_{L^\infty}$. The counterexample to (30) in [35] is not in A_1. Essentially, disproving (31) amounts to finding a "smooth" bad weight, which is even more difficult to build than the weight of Reguera–Thiele. While no explicit example has been constructed, the existence of such a weight has been proved using Bellman function techniques. In fact, in [18] it was shown that there exist weights in A_1 such that

$$\|H\|_{L^1(w)\to L^{1,\infty}(w)} \ge c\, [w]_{A_1} \log^{1/5} [w]_{A_1}\,. \tag{32}$$

6. Some loglog-bumps

Consider the next level of complexity for bump functions, namely: consider the special case when A and B are "loglog-bumps": that is, of the form

$$A(t) = t^p \log^{p-1}(e+t) \log\log^{p-1+\delta}(e^e+t) \quad \bar{A}(t) \approx \frac{t^{p'}}{\log(e+t)\log\log^{1+\delta'}(e^e+t)}\,, \tag{33}$$

$$B(t) = t^{p'} \log^{p'-1}(e+t) \log\log^{p'-1+\delta}(e^e+t), \qquad \bar{B}(t) \approx \frac{t^{p}}{\log(e+t)\log\log^{1+\delta''}(e^e+t)}\,, \tag{34}$$

where $\delta > 0$.

Recall that in the previous section we worked with

$$\sup_Q \langle u\rangle_Q^{1/p} \|\sigma^{1/p'}\|_{B,Q} < \infty, \tag{35}$$

$$\sup_Q \|u^{1/p}\|_{A,Q} \langle \sigma\rangle_Q^{1/p'} < \infty. \tag{36}$$

Now we want to work with a more standard bump condition:

$$\sup_Q \|u^{1/p}\|_{A,Q} \|\sigma^{1/p'}\|_{B,Q} < \infty\,, \tag{37}$$

which is stronger than the previous two together, but this is still unknown whether it is sufficient for the boundedness of an arbitrary Calderón–Zygmund operator and arbitrary $p \in (1,\infty)$.

Remark 10. *For $p = 2$ this condition* **is** *sufficient for the boundedness of an arbitrary Calderón–Zygmund operator, see* [27].

For arbitrary p we had to assume in the previous section that "bump" functions A, B are logarithmic, see (5), (6). Condition on weights was (35) and (36). Now we will assume a stronger condition on weights, namely (37), but we relax the assumptions on A, B. We will assume that they are loglog-bump functions as in (33), (34).

Theorem 6.1. *Given p, $1 < p < \infty$, suppose the pair of gauge functions A, B satisfies loglog-bump conditions* (33), (34) *with* **sufficiently large positive** *δ, and the pair of weights (u, v) satisfies* (37). *Given any dyadic shift S of complexity (m, n), $\tau = \max(m, n) + 1$, $\|S(f\sigma)\|_{L^p(u)} \le C\tau^2 \|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the suprema in* (37).

Theorem 6.2. *Given p, $1 < p < \infty$, suppose that the gauge function A satisfies loglog-bump condition* (33) *with* **sufficiently large positive** *δ, and weights (u, v) satisfies condition* (37). *Given any dyadic shift S of complexity (m, n), $\|S(f\sigma)\|_{L^{p,\infty}(u)} \le C\tau^2 \|f\|_{L^p(\sigma)}$, where C depends only on the dimension d and the supremum in* (37).

Remark 11. *As the reader can see we require large δ instead of natural (and equivalent to B_p) requirement $\delta > 0$.*

To prove these results we modify the proof of, say, Theorem 2.3 above. First modify Lemma 3.5.

Let B be as in (34), B_0 is also like that with δ replaced by $\delta/2$. Then

$$\|f\|_{B_0,\mu} \le C\|f\|_{B,\mu}\varepsilon(\frac{\|f\|_{L^p(\mu)}}{\|f\|_{B,\mu}})\,, \tag{38}$$

where $\varepsilon(t) = (\log \frac{C}{t})^{-\kappa}$, where $C = C(p,\delta)$ and $\kappa = \kappa(p,\delta)$, and $p\kappa > 1$ if δ is large enough.

The proof is verbatim the same as the proof of Lemma 3.5.

Now we can rewrite

$$\frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p \tag{39}$$

$$\le \langle u\rangle_Q \|\sigma^{1/p'}\|^p_{B_0,Q}\|\sigma^{1/p}\|^p_{\bar{B}_0,Q} \tag{40}$$

$$\le \|u^{1/p}\|^p_{A,Q}\varepsilon^p(\frac{\langle u\rangle_Q^{1/p}}{\|u^{1/p}\|_{A,Q}})\varepsilon^p(\frac{\langle \sigma\rangle_Q^{1/p'}}{\|\sigma^{1/p'}\|_{B,Q}})\|\sigma^{1/p'}\|^p_{B,Q}\|\sigma^{1/p}\|^p_{\bar{B}_0,Q} \tag{41}$$

$$\le (\|u^{1/p}\|_{A,Q}\|\sigma^{1/p'}\|_{B,Q})^p \cdot (\varepsilon(\frac{\langle u\rangle_Q^{1/p}}{\|u^{1/p}\|_{A,Q}})\varepsilon(\frac{\langle \sigma\rangle_Q^{1/p'}}{\|\sigma^{1/p'}\|_{B,Q}}))^p \cdot (\|\sigma^{1/p}\|^p_{\bar{B}_0,Q})\,. \tag{42}$$

Consider the second bracket. The form of ε gives immediately that $\varepsilon(t_1)\varepsilon(t_2) \le \varepsilon(t_1t_2)$ for all small t_1, t_2. So we continue:

$$\frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^p$$
$$\le C\,(\|u^{1/p}\|_{A,Q}\|\sigma^{1/p'}\|_{B,Q})^p \cdot (\varepsilon(\frac{\langle u\rangle_Q^{1/p}\langle \sigma\rangle_Q^{1/p'}}{\|u^{1/p}\|_{A,Q}\|\sigma^{1/p'}\|_{B,Q}}))^p \cdot (\|\sigma^{1/p}\|^p_{\bar{B}_0,Q})$$

Recall $Q \in \mathcal{P}^a$, so $\langle u\rangle_Q^{1/p}\langle \sigma\rangle_Q^{1/p'} \asymp 2^a$, also $\|u^{1/p}\|_{A,Q}\|\sigma^{1/p'}\|_{B,Q} \asymp 2^b, b \ge a, b \le b_0$. Here b_0 is logarithm of the supremum in (37) plus 1. We should sum up for $a = b_0, b_0-1, \dots, 0, -1, \dots, -\infty$. Of course it is enough to sum up over all negative a. Let us estimate the product of the first two terms in the last estimate. If b is negative and $|b| \ge |a|/2$ then the first term is bounded by $2^{-\frac{p|a|}{2}}$, and the second is just bounded. If b is positive, or if it is negative but $|b| \le \frac{|a|}{2}$, then we just estimate the first bracket by constant 2^{b_0}, and the argument in the second bracket becomes at most $2^{-\frac{|a|}{2}}$. Hence the second bracket itself is bounded by $\frac{C}{|a|^{p\kappa}}$. As $p\kappa > 1$ the series $\sum_{a:a\,\text{is negative}}[2^{\frac{pa}{2}} + \frac{C}{|a|^{p\kappa}}]$ converges, we can finish the proof of Theorem 6.1 exactly as this has been done in Section 3.

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